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Ideal Graphs Supported By Given Ideals of Commutative Rings

F. H. Abdulqadr[1*]

[1] *Mathematics Department, College of Education, Salahaddin University, Erbil-Iraq*
**Corresponding author's e-mail: fryad.abdulqadr@su.edu.kr*

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Abstract

In this paper we introduce and study a new kind of graph that constructed by non-trivial ideals of a commutative ring with identity. Let R be a commutative ring with identity and P be a non-trivial ideal of R. The ideal graph supported by the ideal P, denoted by $G_R(P)$, is the undirected graph whose vertices are those non-trivial ideals I of R such that there exists a non-trivial ideal J≠I of R with IJ⊂P, and every two vertices I and J are adjacent if I≠J and IJ⊂P. We investigate the connectivity, completeness and planarity of the graph $G_R(P)$. Also we explore the diameter, girth, domination, clique number and chromatic number of $G_R(P)$.

1. Introduction

In the recent years, many kinds of graphs associated with a ring were introduced. In 1988, the zero divisor graph of commutative rings was introduced by Beck I. [3], but his motive was in graph coloring. After that, it has been studied extensively by several authors, see references [1, 2, 7, 8, 9, 10]. This concept was generalized in [4] to the annihilating ideal graph associated to commutative rings, which was introduced by Behboodi M. in 2011 as a graph whose vertices are non-trivial ideals of R, and two distinct vertices I and J are adjacent if IJ=(0).

In this paper, we introduce the concept of the ideal graph supported by a non-trivial ideal of a commutative ring R with identity to extend the idea of the annihilating ideal graph of R, since the statement IJ=(0) associated as a part of the statement IJ⊂P.

The rings considered through this work, are commutative with identity and we use R, Min(R) and Max(R) to denote the ring, the set of minimal and maximal ideals of R respectively. Also we use V(G), $\omega(G)$, $\chi(G)$ and $g(G)$ to denote the vertex set, the clique number, the chromatic number and the girth of a graph G.

Recall that a graph G(V, E) is connected if there is a path between every two of its distinct vertices. For distinct vertices x and y of G, let d(x,y) be the length of the shortest path between x and y. The diameter of G is diam(G)=$\mathrm{Max}_{x,y\in V(G)}\, d(x,y)$. The eccentricity of a vertex v of G is e(v)= $\mathrm{Max}_{x\in V(G)}\, d(x,v)$. The radius of a graph G is rad(G)= $\mathrm{Min}_{x\in V(G)}\, e(x)$. A vertex v is said to be central of G if e(v)=rad(G). The girth of G is the length of the shortest cycle in G. A graph in which each pair of distinct vertices are adjacent is called a complete graph. A ring R is called left (right) Artinian if the set of all the left (right) ideals of R satisfies the descending chain condition, that is, each strictly descending chain of left (right) ideals is finite.

2. The graph supported by an ideal of commutative rings

In this section, we introduce the notion of the ideal graph supported by the ideals of a ring R, and we explore some of its properties and characterizations.

Definition2.1: Let R be a commutative ring with identity and P be a non-trivial ideal of R. The ideal graph supported by the ideal P, denoted by $G_R(P)$, is a graph whose vertices are those non-trivial ideals I of R such that there exists a non-trivial ideal J≠I of R with IJ⊂P, and every two vertices I and J are adjacent if I≠J and IJ⊂P.

Before starting our results, we give the following example.

Example1: Consider the ring of integers z_{24} modulo 24. The following figure shows the graph $G_{z_{24}}((6))$.

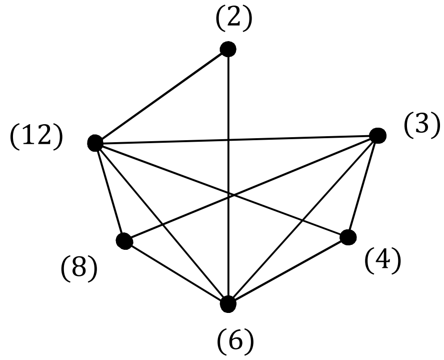

Figure1. The graph $G_{z_{24}}((6))$.

We begin this section with the following easy result which may be needed in the sequel.

Lemma2.2:

1. If P and Q are non-trivial ideals of R such that Q⊆P, then $G_R(Q)$ is a subgraph of $G_R(P)$.
2. If I∈V(G_R(P)) such that I⊂P, then I is adjacent to each other vertices in G_R(P).
3. Every two minimal ideals of R are adjacent vertices in G_R(P).

Proof: The proof is trivial.

The next result shows the completeness of G_R(P).

Propositio2.3: If Max(R)={P}, then G_R(P) is a complete graph.

Proof: The proof follows from the second part of Lemma2.2.

In general, if P∈Max(R) with $|Max(R)| > 1$, the graph G_R(P) may not be complete. We illustrate it in the following example.

Example2: Consider a ring z_{30}. The graph $G_{z_{30}}((3))$ that is shown below, is incomplete.

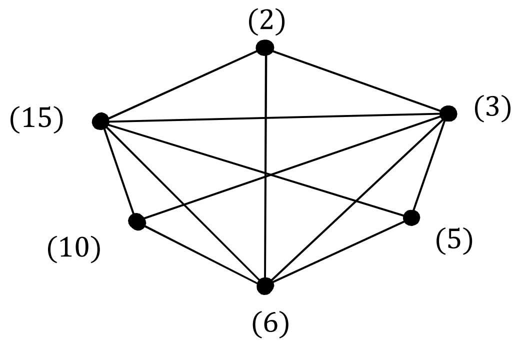

Figure2. The graph $G_{z_{30}}((3))$

In the next result, we demonstrate the adjacency between the minimal ideal vertices of G_R(P) with the other vertices.

Theorem2.4: Let R be an Artinian ring. Then every I∈ V(G_R(P))\Min(R) is adjacent to at least one minimal ideal of R.

Proof: Let I∈ V(G_R(P))\Min(R). Then there exists J∈ V(G_R(P))-I such that IJ⊂P. First, we assume that P∉Min(R). Since R is Artinian, P contains properly a minimal ideal, say L. It follows from Lemma2.2 that I is adjacent to the minimal ideal L in G_R(P). Now, suppose that P∈Min(R). If J∈Min(R), the proof will be completed. Otherwise, J contains an N∈Min(R). This gives that IN⊆IJ⊂P. Thus I is adjacent to N in G_R(P).

Corollary2.5: The domination of G_R(P) is at most |Min(R)|.

Proof: The proof follows from Theorem2.4.

Next, we turn to the following result.

Proposition2.6: If P∈Min(R) and I ∈ V(G_R(P)), then every x∈I\{0} is a zero divisor of R.

Proof: Let I ∈ V(G_R(P)). From the definition of G_R(P), IJ⊂P for some J∈ V(G_R(P))-I. Since P∈Min(R), then IJ=(0). This ends the proof.

In the next result, we find a lower bound of the clique number of $G_R(P)$.

Theorem2.7: If $P\notin Min(R)$ is a non-idempotent ideal, then the clique number of $G_R(P)$ is greater than or equal to $|\{aP: a\in R\backslash Ann(P)\}|$.

Proof: Let I_1, $I_2\in\{aP: a\in R\backslash Ann(P)\}$ with $I_1\neq I_2$. Then $I_1=bP$ and $I_2=cP$, for some $b, c\notin ANN(P)$. Since $I_1I_2\subseteq P^2\subseteq P$ and P is not idempotent, then $I_1I_2\subset P$. This means that I_1 and I_2 are adjacent in $G_R(P)$. Thus the induced subgraph of $G_R(P)$ by $\{aP: a\in R\backslash Ann(P)\}$ is a complete graph. Hence $\omega(G_R(P))\geq|\{aP: a\in R\backslash Ann(P)\}|$.

In the next main result, we investigate the connectivity and the girth of $G_R(P)$.

Theorem2.8:

1. The graph $G_R(P)$ is connected and $diamG_R(P)\leq 3$.
2. If $P\notin Min(R)$ and $G_R(P)$ contains a cycle, then $g(G_R(P))=3$.

Proof:

1. Let I, $J\in V(G_R(P))$ with $I\neq J$. If I and J are adjacent in $G_R(P)$, then thy are connected in $G_R(P)$. Assume that I and J are not adjacent in $G_R(P)$. Then {I, S}, {J, T} are edges of $G_R(P)$, for some S, $T\in V(G_R(P))$. If S=T, then P_2: I, S, J is a path in $G_R(P)$ of length 2. Suppose that $S\neq T$. If $ST\subset P$, then P_3: I, S, T, J is a path in $G_R(P)$ of length 3. Suppose that $ST\not\subset P$. Then $ST\neq(0)$. Obviously, I(ST), $J(ST)\subset P$. Since I and J are not adjacent in $G_R(P)$, then $ST\neq I$, J. Thus P'_2: I, ST, J is a path in $G_R(P)$ of length 2. From each case, we have shown that I and J are connected by a path of length at most 3. Thus $G_R(P)$ is a connected graph and $diamG_R(P)\leq 3$.

2. Since $P\notin Min(R)$, P contains a non-zero ideal, let be L. By Lemma2.2, L is adjacent to every vertex of $G_R(P)$. Since $G_R(P)$ contains a cycle, there exists an edge {I,J} in $G_R(P)$ such that C: L, I, J, L is a cycle in $G_R(P)$. Thus $g(G_R(P))=3$.

In the following example shows that $G_R(P)$ may not have any cycle, when $P\in Min(R)$.

Example3: Consider the ring of integers z_{12} modulo 12.

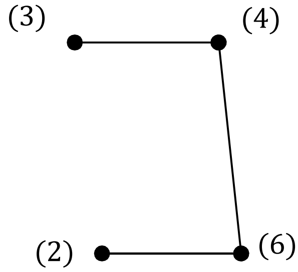

Figure3. The graph $G_{z_{12}}((4))$

In the next result, we find the diameter of $G_R(P)$.

Proposition2.9: If $\{I^k: k=1, 2, \ldots, n\}$ represents the set of all non-trivial ideals of R, then:

$$diamG_R(I^k)=\begin{cases} 1 & , \text{ if } k\in\{1,2\} \\ 2 & , \text{if } k\in\{3,4,\ldots,n\} \end{cases}.$$

Proof: If $k\in\{1, 2\}$, then $I^iI^j\subset I^k$, for every $i\neq j$. It follows that the graph $G_R(I^k)$ is complete. In this case $diamG_R(I^k)=1$. If $k\in\{3,4,\ldots,n\}$, then $I^kI^j\subset I^k$, for every $j\in\{1, 2, \ldots, n\}\backslash\{k\}$. Thus I^k is adjacent to each vertex of $G_R(I^k)$. This yields that $diamG_R(I^s)\leq 2$. On the other hand $I^2I\not\subset I^k$. Thus $d(I^2, I)>1$. Hence $diamG_R(I^k)=2$.

The next result shows central vertices of $G_R(P)$.

Proposition2.10: For a prime ideal P of R, every $I\in V(G_R(P))\backslash\{P\}$ that is not adjacent to P, is a central vertex of $G_R(P)$.

Proof: Let $I\in V(G_R(P))\backslash\{P\}$ not adjacent to P. Then $IJ\subset P$, for some $J\in V(G_R(P))\backslash\{I, P\}$. Since P is prime, then either $I\subset P$ or $J\subset P$. It follows from the second part of Lemma2.2 that either I or J is a central ideal vertex of $G_R(P)$.

The next result shows the upper bound of radius of $G_R(P)$.

Proposition2.11: The $rad(G_R(P))\leq 2$.

Proof: The proof follows from Theorem2.12 and the second part of Lemma2.2.

The following result shows the necessary condition for a vertex of $G_R(P)$ to be cut-vertex.

Theorem2.12: Every cut-vertices of $G_R(P)$ is a minimal ideal of R.

Proof: Let $T\in V(G_R(P))$ be a cut-vertex. Then $TS\subset P$, for some $S\in V(G_R(P))$-T. Now, assume that $T\notin Min(R)$, then T contains properly a non-zero ideal L. If L=S, then $L\in V(G_R(P))$. If $L\neq S$, then $LS\subseteq TS\subset P$ which leads to $L\in V(G_R(P))$. Suppose that K is a vertex adjacent to T in $G_R(P)$ such that S belongs to a component of $G_R(P)$-{T} not containing L. This gives that $LK\subseteq TK\subset P$. Thus {L, K} is an edge in $G_R(P)$. This contradicts that K and L are in different components of $G_R(P)$-{T}. Therefore, T must be a minimal ideal of R.

In general, the converse of Theorem2.11 may not be true. We illustrate it by the following example.

Example4: The following figure shows the graph $G_{z_{32}}((8))$.

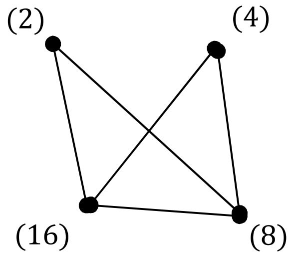

Figure4. The graph $G_{z_{32}}((8))$

Obviously, (16) is a minimal ideal of z_{32} but it is not a cut-vertex of $G_{z_{32}}((8))$.

In the next result, we show that $G_R(P)$ has no cut-vertex, when P is a maximal ideal of R.

Theorem2.13: If $P\in Max(R)$, then $G_R(P)$ is a block.

Proof: Assume that $G_R(P)$ is not block. Then $G_R(P)$ has at least one cut-vertex, let be I. Suppose that L_1 and L_2 are two ideal vertices in two different components of $G_R(P)$-{I}. Since $P\in Max(R)$, then $P\neq I$ by Theorem2.11. If $P\in\{L_1, L_2\}$, then $L_1, L_2\subseteq P$. The maximally of P gives that $L_1, L_2\subset P$. If $P\notin\{L_1, L_2\}$, then $L_1P, L_2P\subset P$. Thus P_2: L_1, P, L_2 is a path in $G_R(P)$. Both cases contradict that L_1 and L_2 are two ideal vertices in two different components of $G_R(P)$-{I}. Therefore, $G_R(P)$ must be a block.

3. The graph $G_{z_{q^n}}((q^m))$

In this section, we explore some properties of $G_{z_{q^n}}((q^m))$ of a ring of integers z_{q^n} modulo q^n, where q is a prime number, and m and n are integers with $1\leq m<n$ and $n>2$.

We begin this section with the following main result.

Proposition3.1: The graph $G_{z_{q^n}}((q^m))$is complete if and only if either m=1 or m=2.

Proof: The proof follows from the definition of the graph $G_{z_{q^n}}((q^m))$.

In the next result, we find the radius and central of $G_{z_{q^n}}((q^m))$.

Proposition3.2: The radius of $G_{z_{q^n}}((q^m))$ is equal to 1, and the central vertices of $G_{z_{q^n}}((q^m))$ are (q^m), (q^{m+1}), …, (q^{n-1}), when m>2.

Proof: Since $(q^b)(q^{n-1})\subset(q^m)$, for every $b\in\{1, 2, \dots, n-1\}$, then every vertex (q^b) is adjacent to (q^{n-1}) in $G_{z_{q^n}}((q^m))$. Thus the eccentricity of (q^{n-1}) is equal to 1. This gives that the radius of $G_{z_{q^n}}((q^m))$ is equal to 1. Suppose that m>2 and a is any positive integer with a<n. If $a\geq m$, then $(q^a)(q^r)\subset(q^m)$, for every r=1, 2, …, $n-1$ with $r\neq a$. In this case $e((q^a))=1$. Assume that a<m. Obviously, $(q^a)(q^2)\not\subset(q^m)$, when a=1, and $(q^a)(q)\not\subset(q^m)$, when a>1. Thus the eccentricity of (q^a) is greater than 1. From both cases, the radius of $G_{z_{q^n}}((q^m))$ is equal to 1, and the central vertices of $G_{z_{q^n}}((q^m))$ are $(q^m), (q^{m+1}), \dots, (q^{n-1})$.

The next result demonstrates the relationship between the ideal graphs supported by two distinct ideals of z_{p^n}.

Theorem3.3: Let m and p be two integers with $1\leq m, p<n$ The graphs $G_{z_{q^n}}((q^m))$ and $G_{z_{q^n}}((q^p))$ are identical if and only if they are complete graphs.

Proof: It is easy to show $G_{z_{q^n}}((q^m))$ and $G_{z_{q^n}}((q^p))$ are identical, when they are complete. Now, suppose that $G_{z_{q^n}}((q^m))$ and $G_{z_{q^n}}((q^p))$ are identical. Assume by contrary that at least one of the graphs $G_{z_{q^n}}((q^m))$ and $G_{z_{q^n}}((q^p))$ is not complete, let be $G_{z_{q^n}}((q^m))$. Then by Proposition3.1, m>2. If p $\in\{1, 2\}$, then $G_{z_{q^n}}((q^p))$ is a complete graph. Let that p>2. Clearly, $G_{z_{q^n}}((q^m))$ and $G_{z_{q^n}}((q^p))$ have the same vertex set including the maximal ideal (q). Obviously, all adjacent vertices to (q) are $\{(q^m), (q^{m+1}), (q^{m+2}), \ldots, (q^{n-1})\}$ and $\{(q^p), (q^{p+1}), (q^{p+2}), \ldots, (q^{n-1})\}$ in $G_{z_{q^n}}((q^m))$ and $G_{z_{q^n}}((q^p))$ respectively. This shows that $\deg_{G_{z_{q^n}}((q^m))}(q) \neq \deg_{G_{z_{q^n}}((q^p))}(q)$. Both cases contradict the fact that $G_{z_{q^n}}((q^m))$ and $G_{z_{q^n}}((q^p))$ are identical graphs. Therefore, $G_{z_{q^n}}((q^m))$ and $G_{z_{q^n}}((q^p))$ are complete graphs.

Example5: Consider the ring of integers z_{64} modulo 64. The following figure shows that $G_{z_{64}}((2))$ is a complete graph.

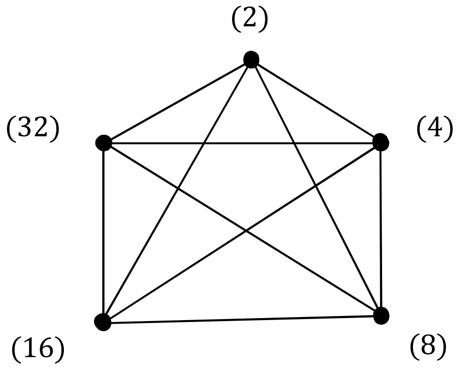

Figure5. The graph $G_{z_{64}}((2))$

If we draw the graph $G_{z_{64}}((4))$, we get the same graph as in Figure5. Hence $G_{z_{64}}((2))$ and $G_{z_{64}}((4))$ are identical graphs.

In the next main result, we find the clique number of $\chi_{(p^s)}(z_{p^n})$.

Theorem3.4: Let n>3. The clique and chromatic numbers of $G_{z_{q^n}}((q^m))$ are $\omega(\chi_{(p^s)}(z_{p^n}))=n-\lceil \frac{m}{2} \rceil$ and $\chi(G_{z_{q^n}}((q^m))) = n - \lceil \frac{m}{2} \rceil + 1$ respectively.

Proof: Let m$\in\{1, 2\}$. Then by Proposition3.1, the graph $G_{z_{q^n}}((q^m))$ is a complete graph. In this case, $\omega(G_{z_{q^n}}((q^m)))=\chi(G_{z_{q^n}}((q^m))) = n-1$. In general, for m$\in\{r, r+1\}$ with r is an odd positive integer less than n, the induced subgraph of $G_{z_{q^n}}((q^m))$ by the vertex set $W=\{(q^{\lceil \frac{r}{2} \rceil}), (q^{\lceil \frac{r}{2} \rceil+1}), (q^{\lceil \frac{r}{2} \rceil+2}), \ldots, (q^{n-1})\}$ is complete. To show that W is a maximal complete subgraph of $G_{z_{q^n}}((q^m))$, let I$\in G_{z_{q^n}}((q^m))\backslash W$. Then $I=(q^s)$ for some $s\leq \lceil \frac{r}{2} \rceil - 1$. Obviously, $s+\lceil \frac{r}{2} \rceil \leq 2\lceil \frac{r}{2} \rceil - 1 \leq m$, then $(q^s)(q^{\lceil \frac{r}{2} \rceil})\not\subset(q^r)$. This means that a graph obtaining by adding any vertex of W, will not be complete. Therefore, W is a maximal complete subgraph of $G_{z_{q^n}}((q^m)$. Thus $\omega(G_{z_{q^n}}((q^m)))=(n-1) - (\lceil \frac{r}{2} \rceil - 1)$. Since r is odd, then $\omega(G_{z_{q^n}}((q^m)))=n-\lceil \frac{m}{2} \rceil$. Obviously, the graph induced by W is $n-\lceil \frac{r}{2} \rceil$-colorable. On the other hand, the vertices (q^α) and (q^β) are not adjacent in $G_{z_{q^n}}((q^m))$, for every $\alpha, \beta< \lceil \frac{r}{2} \rceil$. So, we can color every vertex (q^α) in which $\alpha<\lceil \frac{r}{2} \rceil$ in the same color. Thus the chromatic number of $G_{z_{q^n}}((q^m))$ is $\chi(G_{z_{q^n}}((q^m))= \omega(G_{z_{q^n}}((q^m)))+1=n-\lceil \frac{m}{2} \rceil + 1$.

The next result illustrates the planarity of $G_{z_{q^n}}((q^m))$.

Theorem3.5: Let $n > 6$. Then the graph $G_{z_{q^n}}((q^m))$ is planar if and only if (n, m)$\in\{(7, 5), (7, 6), (8, 7)\}$.

Proof: Suppose that(n, m)$\in\{(7, 5), (7, 6), (8, 7)\}$. First, we show that $G_{z_{q^n}}((q^m))$ does not contain any complete bipartite graph K(3, 3). Assume by contrary that $G_{z_{q^n}}((q^m))$ contains a K(3, 3) with partite sets $V=\{(q^{a_i})$: i=1, 2, 3$\}$ and $W=\{(q^{b_i})$: i=1, 2, 3$\}$. Since $(q^{a_i})(q^{b_j})\subset(q^m)$, for every i, j=1, 2, 3, then $(a_i+b_j)>m$. On the other hand $(q^{a_i})(q^{a_j})$, $(q^{b_i})(q^{b_j})\not\subseteq(q^m)$, for every i, j=1, 2, 3 with i$\neq$j, then a_i+a_j, $b_i+b_j\leq m$. Thus

$(a_i+b_i)+(a_j+b_j) \leq 2m$. This contradicts that $a_i+b_j>m$. Hence $G_{Z_{q^n}}((q^m))$ does not contain a complete bipartite subgraph. If (n, m)=(8, 7), then $G_{Z_{q^n}}((q^m))$ is constructed as follows:

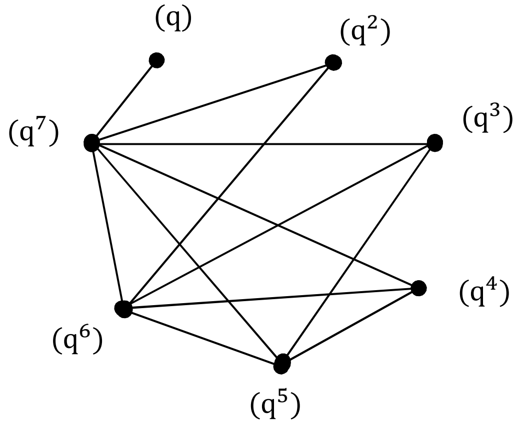

Figure6. The graph $G_{Z_{q^8}}((q^7)$

Obviously, $G_{Z_{q^8}}((q^7))$ does not contain K_5. Thus, by Koratowsky theorem (See [6]), the graph $G_{Z_{q^n}}((q^m))$ is a planar graph. Similarly, we can prove that $G_{Z_{q^n}}((q^m))$ is a planar graph, when (n, m)∈{(7, 5), (7, 6)}. Conversely, suppose that $G_{Z_{q^n}}((q^m))$ is a planar graph. We have to show that (n, m)∈{(7, 5), (7, 6), (8, 7)}. Assume that (n, m)∉{(7, 5), (7, 6), (8, 7)}. If m∈{1, 2}, then Proposition3.1, $G_{Z_{q^n}}((q^m))$ is a non-planar graph. Let m>2. If n=7, then m∈{3, 4} and hence $G_{Z_{q^n}}((q^m))$ contains a complete subgraph whose vertices are (q^2), (q^3), (q^4), (q^5) and (q^6). If n=8, then m∈{3, 4, 5, 6}. Similarly, we can find a complete subgraph of $G_{Z_{q^n}}((q^m))$ of order 5. Suppose that n≥9. Obviously, $(q^{n-i})\,(q^{n-j}) \subset (q^m)$, for every i, j=1, 2, 3, 4, 5 with i≠j. Thus $G_{Z_{q^n}}((q^m))$ contains a complete subgraph whose vertices are (q^{n-1}), (q^{n-2}), (q^{n-3}), (q^{n-4}) and (q^{n-5}). Each case contradicts the fact that $G_{Z_{q^n}}((q^m))$ is a planar graph. Therefore (n, m) ∈{(7, 5), (7, 6), (8, 7)}.

References

[1] Akbari, S. and Mohamadian, A., *"On the Zero-Divisor Graph of a Commutative Ring"*, Journal of Algebra. Vol. 274, No. 2, pp.847-855. (2004).

[2] Andersn, D. F. and Livingston, P. S., *"The zero divisor graph of a commutative ring",* Journal of Algebra. Vol. 217, pp. 434-447. (1999).

[3] Beck, I., *"Coloring of commutative rings"*, Journal of Algebra. 116, pp. 208–226. (1988).

[4] Behboodi, M. and Rakeei, Z., *"The Annihilating-Ideal Graph of Commutative Rings I"*, Journal of Algebra and Appl., Vol.10, No.4, pp. 727-739. (2011).

[5] David, S. and Richard, M., *"Abstract Algebra"*, Prentice-Hall Inc. U. S. A. (1991).

[6] Gary, C. and Linda, L., *"Graphs and Digraphs"*, 2nd ed., Wadsworth and Brooks/Cole, California. (1986).

[7] Gupta, R, Sen, S. M. K. and Ghosh, S., *" A variation of zero-divisor graphs"*, Discuss. Math. Gen. Algebra Appl. Vol. 35**,** No. 2, pp. 159–176. (2015).

[8] Mahadevi, P. and Babujee, J., " *On Special Structures of Zero-Divisor Graphs*", International Journal of Pure and Applied Mathematics. Vol. 119, No. 13, pp. 281-287. (2018)

[9] Patil, A., Waphare, B. N. and Joshi, V., "*Perfect zero-divisor graphs*", Discrete Math. Vol. 340, No. 4, pp. 740–745. (2017)

[10] Shuker, N. H., Mohammad, H. Q. and Luma, A. K., *"Hosoya and Wiener Index of Zero-Divisor Graph of $Z_{p^m q^2}$,"Zanco Journal of Pure and Applied Sciences*. Vol. 31, No. 2, pp. 45-52. (2019).